Hydrodynamic limits and stationary fluctuations for the facilitated exclusion process

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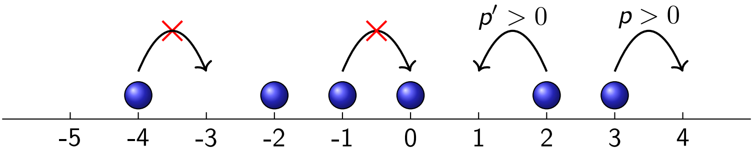
Facilitated exclusion processes

- Exclusion rule: there is at most one particle at each site.
- Facilitated rule: a particle has to be pushed forward in order to jump.
- For each $x \in \mathbb{Z}$, $\eta_x(t) \in \{0,1\}$. Thus, the configuration

$$\eta(t) = \{\eta_x(t)\}_{x\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}.$$

The process $\{\eta(t)\}$ is a continuous-time Markov process on $\{0,1\}^{\mathbb{Z}}$.

- Symmetric: $p = p'$. Asymmetric: $p > p'$. Weakly asymmetric: $p_N - p'_N \to 0$.



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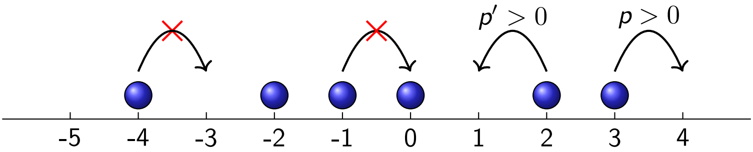

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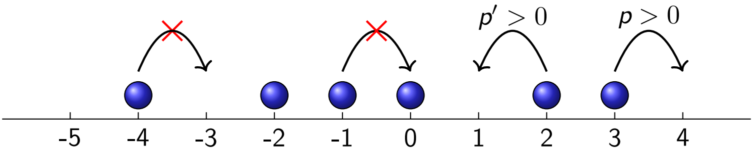

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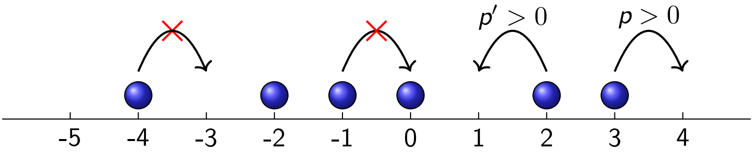

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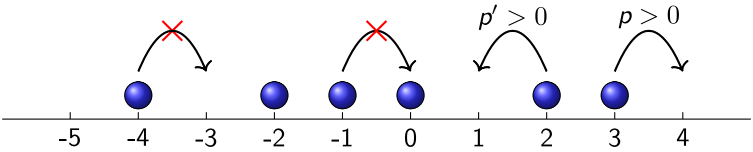

Phase Transitions

The process has degenerate rates, and displays a phase transition at the critical particle density $\rho_c = 1/2$.

- If initially $\rho > 1/2$, then the system evolves until there are no longer two neighboring empty sites.

$t = 0$

$t > 0$:ergodic

- If initially $\rho < 1/2$, then the system evolves until all particles are isolated, and thus can no longer move.

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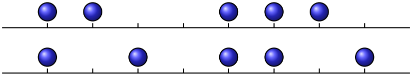

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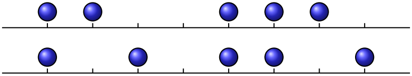

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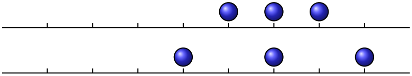

Invariant measures

- For each $\rho > 1/2$, the process has a translation invariant and invariant measure π_ρ, which is not product. (exponentially decay)

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- Put a hole in some position with probability $1-\rho$, then put a random geometric number of parameter $\frac{1-\rho}{\rho}$ particles to its right, then a hole, starts again and so on.

$$\pi_\rho(11) = \pi_\rho(1) - \pi_\rho(01) = \rho - (1-\rho) \times 1 = 2\rho - 1.$$

$$\pi_\rho(111) = \pi_\rho(11) - \pi_\rho(011)$$

$$= (2\rho - 1) - (1-\rho) \times 1 \times \frac{2\rho - 1}{\rho} = \frac{(2\rho-1)^2}{\rho}.$$

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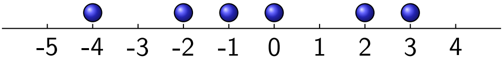
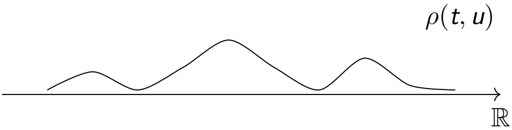
Hydrodynamic limits

- Concerns macroscopic behavior of interacting particle systems.
- Prove LLN for the empirical measure of the process

$$\pi_t^N(du) = \frac{1}{N} \sum_{x \in \mathbb{Z}} \eta_x(t\, N^a)\delta_{x/N}(du),$$

whose limit is usually given by some PDE (hydrodynamic equation).
- Entropy method (郭懋正, Papanicolau, Varadhan), relative entropy method (姚鸿泽), attractiveness method etc. Cf. [Kipnis&Landim'98].





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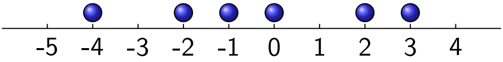

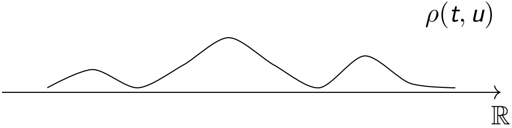

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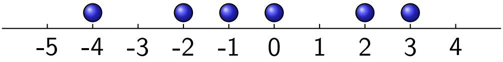

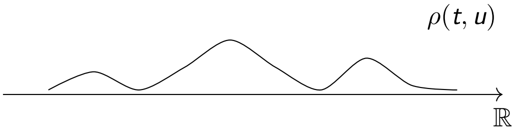

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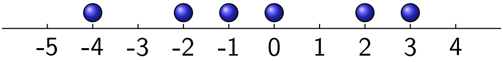

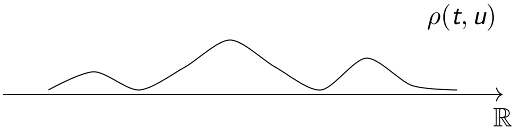

Theorem [Blondel *et al.* '20, '21, Erignoux, Simon and Z. '22]

Assume $\pi_0^N(du) \to \rho^{\rm ini}(u)du$ in probability, as $N \to \infty$, for some profile $\rho^{\rm ini} : \mathbb{R} \to [0,1]$. Take

$$a = \begin{cases} 2, & \text{if } p = p' = 1; \\ 1, & \text{if } p \neq p'. \end{cases}$$

Then, for any $t > 0$, $\pi_t^N(du) \to \rho(t,u)du$ in probability, where
(i) for $p = p' = 1$,

$$\partial_t \rho(t,u) = \partial_u^2 \Big(\frac{2\rho(t,u) - 1}{\rho(t,u)} \mathbf{1}_{\{\rho(t,u) > 1/2\}} \Big),$$

(ii) for $p \neq p'$, (entropy solution)

$$\partial_t \rho(t,u) + (p - p')\partial_u \Big(\frac{(1-\rho(t,u))(2\rho(t,u)-1)}{\rho(t,u)} \mathbf{1}_{\{\rho(t,u) > 1/2\}} \Big) = 0.$$

How to see the hydrodynamic equation?

Consider $p = p' = 1$. Remember, for $u \in \mathbb{R}$,

$$\rho(t, u) \approx \eta_{uN}(tN^2) \approx \mathbb{E}[\eta_{uN}(tN^2)].$$

Then,

$$\partial_t \rho(t, u) \approx \mathbb{E}[(N^2 L)\eta_{[uN]}(tN^2)].$$

By conservation laws,

$$L\eta_0 = j_{-1,0}(\eta) - j_{0,1}(\eta),$$
$$j_{-1,0}(\eta) = \eta_{-2}\eta_{-1}(1 - \eta_0) - \eta_1\eta_0(1 - \eta_{-1}).$$

The FEP is gradient in the sense that

$$j_{-1,0}(\eta) = h_{-1}(\eta) - h_0(\eta),$$
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Therefore,

$$\partial_t \rho(t,u) \approx N^2 \{ \mathbb{E}[h_{uN+1}] + \mathbb{E}[h_{uN-1}] - 2\mathbb{E}[h_{uN}] \}.$$

Note that

$$E_{\pi_\rho}[h_0(\eta)] = \frac{2\rho - 1}{\rho} \mathbf{1}_{\rho > 1/2} =: \varphi(\rho).$$

Finally,

$$\begin{aligned} \partial_t \rho(t,u) &\approx N^2 \{ \varphi(\rho(t, u + \tfrac{1}{N})) + \varphi(\rho(t, u - \tfrac{1}{N})) - 2\varphi(\rho(t,u)) \} \\ &\approx \Delta \varphi(\rho(t,u)). \end{aligned}$$

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Stationary fluctuations

- Let the initial measure of the process be π_ρ for some $\rho \in (1/2, 1)$.
- Consider $p = 1 + N^{-\kappa}$ for some $\kappa \geq 0$, and $p' = 1$.
- Define the density fluctuation fields

$$Y_t^N(G) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} \big(\eta_x(tN^a) - \rho\big) G\big(\tfrac{x}{N} - tvN^{a-\kappa-1}\big),$$

where

$$v(\rho) := \frac{d}{d\rho} E_{\pi_\rho}[\eta_{x-1}\eta_x(1-\eta_{x+1})] = \frac{d}{d\rho} \frac{(1-\rho)(2\rho-1)}{\rho} = \frac{1-2\rho^2}{\rho^2}.$$

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KPZ equation

The Kardar-Parisi-Zhang (KPZ) equation is

$$\partial_t h = \nu \partial_u^2 h + \lambda (\partial_u h)^2 + \sqrt{D} \dot{\mathcal{W}}_t,$$

which describes the random interface growth in physics.

- Well posedness via regularity structures [Hairer, Ann. of Math'13, Invent. Math.'14], and paracontrolled distributions theory [Gubinelli, Imkeller and Perkowski, Forum of Mathematics'15];
- Cole-hopf solution ($Z = e^h$) via the study of some integrable stochastic systems [Bertini & Giacomin'97, Commun. Math. Phys];
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Let $Y_t = \partial_u h_t$, we get the stochastic Burgers equations (SBE)

$$\partial_t Y_t = \nu \partial_u^2 Y_t + \lambda \partial_u Y_t^2 + \sqrt{D} \partial_u \dot{\mathcal{W}}_t.$$

Definition

We say the process Y_t satisfies the energy condition if for $0 \le s \le t \le T$,

$$A_{s,t}^{\varepsilon}(G) := \int_s^t \int_{\mathbb{R}} Y_\tau(\iota_\varepsilon)^2 \partial_u G(u) \, du \, d\tau, \quad G \in \mathcal{S}(\mathbb{R})$$

is a Cauchy sequence in $L^2(P)$ as $\varepsilon \to 0$, where $\iota_\varepsilon = (2\varepsilon)^{-1} \mathbf{1}_{(-\varepsilon,\varepsilon)}$. Denote by $A_{s,t}$ the limit of $A_{s,t}^{\varepsilon}$.

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Definition [Goncalves & Jara'14, Arch. Ration. Mech. Anal.]

We say Y_t is a stationary energy condition to the SBE if

1. for any $t > 0$, and $G, H \in \mathcal{S}$,

$$E(Y_t(G)Y_t(H)) = \chi(\rho)\langle G, H\rangle.$$

2. Y satisfies the L^2 energy condition, so that for any $t \geq 0$, the tempered distribution $A_{0,t} \in \mathcal{S}'$ is well-defined.
3. for any $G \in \mathcal{S}$,

$$M_t(G) := Y_t(G) - Y_0(G) - \nu \int_0^t Y_s(\partial_u^2 G)ds + \lambda A_{0,t}(G),$$

$$N_t(G) := \left[M_t(G)\right]^2 - 2tD\|\partial_u G\|^2_{L^2(\mathbb{R})}$$

are both integrable martingales w.r.t. Y's natural filtration.

For uniqueness of the energy solution, cf. [Gubinelli, & Perkowski'18, J. Amer. Math. Soc.].

Recall $p - p' = N^{-\kappa}$.

Theorem [Erignoux and Z.'23]

For $\kappa \geq 1/2$ take $a = 2$, and for $\kappa = 0$ take $a < 4/3$. Then,

$$\{Y_t^N, 0 \leq t \leq T\} \Rightarrow \{Y_t, 0 \leq t \leq T\},$$

where

- for $\kappa > 1/2$,

$$\partial_t Y_t = D(\rho)\partial_u^2 Y_t + \sqrt{2\sigma(\rho)}\partial_u \dot{\mathcal{W}}_t$$

- for $\kappa = 1/2$,

$$\partial_t Y_t = D(\rho)\partial_u^2 Y_t + \frac{1}{2}D'(\rho)\partial_u Y_t^2 + \sqrt{2\sigma(\rho)}\partial_u \dot{\mathcal{W}}_t$$

- for $\kappa = 0$,

$$E[Y_t(G)Y_s(H)] = \chi(\rho)\langle H, G\rangle$$

  for all $0 \leq s, t \leq T$ and $H, G \in \mathcal{S}$. (No evolution)

Why $\kappa = 1/2$?

The current is

$$N^{2-\kappa} j_{0,1} = N^{2-\kappa}[h_0 - h_1].$$

The gradient kills one N, and $N^{1-\kappa} h_0$ survives. By second-order Boltzmann-Gibbs principle,

$$\begin{aligned} h_0(\eta) - \varphi(\rho) &\approx \varphi'(\rho)(\eta_0 - \rho) + \frac{1}{2}\varphi''(\rho)(\eta_0 - \rho)^2 \\ &\approx \varphi'(\rho)(\eta_0^{\varepsilon N} - \rho) + \frac{1}{2}\varphi''(\rho)(\eta_0^{\varepsilon N} - \rho)^2. \end{aligned}$$

We need the following term has order one,

$$\frac{N}{\sqrt{N}} N^{1-\kappa} (\eta_0^{\varepsilon N} - \rho)^2 = \mathcal{O}\Big(\frac{1}{2} - \kappa\Big).$$

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Universality

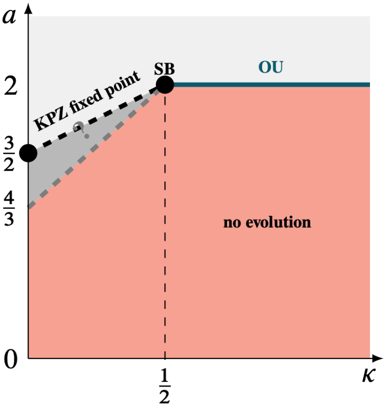

Figure: From [Gonçalves, ICM 2022 proceedings]

Theorem [Matetski, Quastel & Remenik'21, Acta. Math.], [Quastel & Sarkar'23, J. Amer. Math. Soc.]

For asymmetric exclusion process, under suitable initial conditions and under $1:2:3$ scaling, the height function of the process converges to the KPZ fixed point.

For particle system with only one conserved quantity, we observe transition from Edwards–Wilkinson (EW) universality class to the Kardar–Parisi–Zhang (KPZ) universality class.

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Co-authors

Figure: Marielle Simon

Figure: Clément Erignoux

Universality for two conservation laws

$G^2 \setminus G^1$	$\begin{pmatrix} \star & \\ & \bullet \end{pmatrix}$	$\begin{pmatrix} \star & \\ & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbf{0} & \\ & \bullet \end{pmatrix}$	$\begin{pmatrix} \mathbf{0} & \\ & 0 \end{pmatrix}$
$\begin{pmatrix} \bullet & \\ & \star \end{pmatrix}$	(KPZ,KPZ)	(KPZ,KPZ)	($\frac{5}{3}$L,KPZ)	(D,KPZ')
$\begin{pmatrix} 0 & \\ & \star \end{pmatrix}$	(KPZ,KPZ)	(KPZ,KPZ)	($\frac{5}{3}$L,KPZ)	(D,KPZ)
$\begin{pmatrix} \bullet & \\ & \mathbf{0} \end{pmatrix}$	(KPZ,$\frac{5}{3}$L)	(KPZ,$\frac{5}{3}$L)	(GM,GM)	(D,$\frac{3}{2}$L)
$\begin{pmatrix} 0 & \\ & \mathbf{0} \end{pmatrix}$	(KPZ',D)	(KPZ,D)	($\frac{3}{2}$L,D)	(D,D)

Table: From [Popkov, Schmidt and Schütz'15, J. Stat. Phys.].

Fibonacci universality classes

Let $\mathbb{I}_\alpha = \{\beta : G^\alpha_{\beta\beta} \neq 0\}$.

- For α such that $\mathbb{I}_\alpha = \emptyset$, one has $z_\alpha = 2$, and the limit belongs to diffusive universality class.
- If $\alpha \in \mathbb{I}_\alpha$, then $z_\alpha = 3/2$.
  - If there is no diffusive mode $\beta \in \mathbb{I}_\alpha$, we have KPZ universality class;
  - if there is at least one diffusive mode $\beta \in \mathbb{I}_\alpha$, we have modified KPZ universality class.
- If $\alpha \notin \mathbb{I}_\alpha$ and $\mathbb{I}_\alpha \neq \emptyset$, then

$$z_\alpha = \min_{\beta \in \mathbb{I}_\alpha} \{1 + \frac{1}{z_\beta}\}.$$

In particular,

$$\{z_\alpha\} \subset \{3/2, 5/3, \ldots, (\sqrt{5}+1)/2\},$$

the Kepler rations of Fibonacci numbers. (Lévy universality class).

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Rigorous results

- Oscillators of chains. Lévy(3/2) and Lévy(5/3).
  - [Bernardin-Goncalves-Jara'16, ARMA]
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  - [Bernardin-Goncalves-Jara-Simon'18, CMP]
  - [Saito-Sasada-Suda'19, CMP]
  - [Goncalves-Hayashi'23, CMP]
- ABC model. (KPZ,KPZ), (KPZ,D), (D,D).
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Thanks!

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