Moderate deviation principles for the tagged particle in the simple exclusion process

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1. Exclusion processes

2. Main results

3. Outline of the proof
The exclusion process

The state space is $\{0, 1\}^{\mathbb{Z}^d}$. An element of the state space is called a configuration, denoted by $\eta$. For $x \in \mathbb{Z}^d$, $\eta_x \in \{0, 1\}$ is the number of particles at site $x$.

Generator of the process $\eta(t)$: for local functions $f$ on $\{0, 1\}^{\mathbb{Z}^d}$,

$$Lf(\eta) = \sum_{x, y \in \mathbb{Z}^d} p(y - x) \eta_x (1 - \eta_y) [f(\eta^x, y) - f(\eta)],$$

where $\eta^x_{x, y} = \eta_x$ for $z = y$, $= \eta_y$ for $z = x$, and $= \eta_z$ otherwise.

Figure: SSEP: $p = q = 1/2$. ASEP: $p = 1 - q \in (1/2, 1]$. 
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The state space is \( \{0, 1\}^\mathbb{Z}^d \). An element of the state space is called a configuration, denoted by \( \eta \). For \( x \in \mathbb{Z}^d \), \( \eta_x \in \{0, 1\} \) is the number of particles at site \( x \).

Generator of the process \( \eta(t) \): for local functions \( f \) on \( \{0, 1\}^\mathbb{Z}^d \),

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\[\text{Figure: SSEP: } p = q = 1/2. \text{ ASEP: } p = 1 - q \in (1/2, 1).\]
For $\rho \in [0, 1]$, let $\nu_\rho$ be the product measure on $\{0, 1\}^\mathbb{Z}^d$ with marginals

$$\nu_\rho(\eta_x = 1) = \rho, \quad x \in \mathbb{Z}^d.$$ 

It is well known that $\nu_\rho$ is invariant for the exclusion process, see (Liggett’85 and ’99).

Check directly that

$$\int Lf d\nu_\rho = 0.$$
The tagged particle

Let the initial measure of the process be

\[ \nu^*_\rho(\cdot) = \nu_\rho(\cdot | \eta_0 = 1). \]

Denote by \( X(t) \) the position of the tagged particle at time \( t \). The process \( X(t) \) is not Markovian!

Define the environmental process seen from the tagged particle as

\[ \zeta_x(t) = \eta_{X(t)+x}, \quad x \in \mathbb{Z}^d. \]

Since the process \( \eta(t) \) is translation invariant, the process \( \zeta(t) \) is Markovian. Moreover, \( \nu^*_\rho \) is invariant for the process \( \zeta(t) \).
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Related work

Assume $p(\cdot)$ has finite range and the process $\eta(t)$ starts from the initial measure $\nu^\rho$. 

**Law of large numbers**

$$\lim_{t \to \infty} \frac{X(t)}{t} = (1 - \rho) \sum_{x \in \mathbb{Z}^d} xp(x) \quad \text{almost surely},$$

see (Saada’87).

**Central limit theorems**

- For $d = 1$, $p(1) = p(-1) = 1/2$,

$$\frac{X_{tN^2}}{N^{1/2}} \Rightarrow fBM(1/4), \quad N \to +\infty.$$ 

See (Arratia’83) (De Masi-Ferrari’02) (Peligrad-Sethuraman’08).
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In the other cases, the tagged particle is diffusive, and we expect

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See (Kipnis-Varadhan’86) for the symmetric case, (Varadhan’95) mean zero case, (Kipnis’86) ASEP, (Sethuraman-Varadhan-Yau’00) asymmetric case in dimension \( d \geq 3 \), (Komorowski-Landim-Olla’12).

For asymmetric case in \( d \leq 2 \) except ASEP, CLT and invariance principles are open, see (Sethuraman’06).

Large deviations

For SSEP, see (Sethuraman-Varadhan’13); for ASEP, see (Sethuraman-Varadhan’23).
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What is MDP?

Let $X_1, X_2, \ldots$ be independent random variables, and $S_N = \sum_{i=1}^{N} X_i$.

Law of large numbers.

$$S_N/N \rightarrow \mu := E[X_1] \quad \text{almost surely.}$$

Central limit theorems.

$$(S_N - N\mu)/N^{1/2} \Rightarrow N(0, \sigma^2).$$

Large deviations.

$$\log \mathbb{P}(S_N/N = x) \approx -NI(x).$$
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Moderate deviations. For $N^{1/2} \ll a_N \ll N$,

$$\log \mathbb{P}\left( \frac{S_N - N\mu}{a_N} = x \right) \approx -\frac{a_N^2}{N \sigma^2} x^2.$$ 

- First intuition:

$$\mathbb{P}\left( \frac{S_N - N\mu}{a_N} = x \right) \approx \mathbb{P}\left( N(0, \sigma^2) = \frac{a_N x}{\sqrt{N}} \right) \approx \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left\{ -\frac{a_N^2}{N \sigma^2} x^2 \right\}.$$ 

- Second intuition:

$$\mathbb{P}\left( \frac{S_N - N\mu}{a_N} = x \right) = \mathbb{P}\left( \frac{S_N}{N} = \mu + \frac{a_N}{N} x \right) \approx \exp \left\{ -NI\left( \mu + \frac{a_N}{N} x \right) \right\}.$$ 

Since $I(\mu) = I'(\mu) = 0$,

$$I\left( \mu + \frac{a_N}{N} x \right) \approx \frac{a_N^2}{2N^2} I''(\mu)x^2.$$
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One-point MDP for the tagged particle

Consider the SSEP, that is, $d = 1$ and $p(1) = p(-1) = 1/2$. Recall

$$X(t)/t^{1/4} \Rightarrow N(0, \sigma^2), \quad \sigma^2 = \sqrt{2/\pi}(1 - \rho)/\rho.$$ 

Fix $\sqrt{N} \ll a_N \ll N$ and $T > 0$. Define

$$I(\alpha) = -\alpha^2/(2\sqrt{T}\sigma^2), \quad \alpha \in \mathbb{R}.$$ 

**Theorem (Xue-Z. ’23)**

The sequence $\{X(TN^2)/a_N\}_{N \geq 1}$ satisfies the MDP with decay rate $a_N^2/N$ and with rate function $I(\cdot)$. Precisely speaking, for any closed set $C \subset \mathbb{R}$ and for any open set $O \in \mathbb{R}$,

$$\limsup_{N \to \infty} \frac{N}{a_N^2} \log \mathbb{P}(X(TN^2)/a_N \in C) \leq - \inf_{\alpha \in C} I(\alpha),$$

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Sample path MDP for the tagged particle

The fractional Brownian motion \( \{B_{1/4}(t) : t \geq 0\} \) is a Gaussian process with covariance

\[
\text{Cov}(B_{1/4}(t), B_{1/4}(s)) = \frac{1}{2} \left( t^{1/2} + s^{1/2} - |t - s|^{1/2} \right).
\]

It also has the following representation

\[
B_{1/4}(t) = \int_0^t K(t, s) dB(s).
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Let \( \mathcal{H} \) be the set of càdlàg functions \( f : [0, T] \to \mathbb{R} \) such that there exists a function \( h_f \in L^2([0, T]) \) satisfying

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For any càdlàg functions \( f : [0, T] \to \mathbb{R} \), define

\[
I_{\text{path}}(f) = \begin{cases} 
\frac{1}{2} \int_0^T h_f(s)^2 \, ds, & \text{if } f \in \mathcal{H}; \\
+\infty, & \text{otherwise}.
\end{cases}
\]

\( I_{\text{path}} \) is the large deviation rate function of the sequence of processes \( \{B_{1/4}(t)/\sqrt{N} : t \geq 0\}_{N \geq 1} \).

Assume

\[
\sqrt{N \log N} \ll a_N \ll N.
\]

**Theorem (Xue-Z. '23)**

The sequence \( \{X(tN^2)/a_N : 0 \leq t \leq T\}_{N \geq 1} \) satisfies the MDP with decay rate \( a_N^2/N \) and with rate function \( I_{\text{path}}(\cdot) \).
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Intuitive explanation

\[
\mathbb{P} \left( \left\{ \frac{X(tN^2)}{a_N} : 0 \leq t \leq T \right\} = \left\{ x(t) : 0 \leq t \leq T \right\} \right) = \mathbb{P} \left( \left\{ \frac{X(tN^2)}{\sqrt{N}} : 0 \leq t \leq T \right\} = \left\{ \frac{a_N}{\sqrt{N}} x(t) : 0 \leq t \leq T \right\} \right) \\
\approx \mathbb{P} \left( \left\{ \frac{\sqrt{N}}{a_N} B_{1/4}(t) : 0 \leq t \leq T \right\} = \left\{ x(t) : 0 \leq t \leq T \right\} \right) \\
\approx \exp \left\{ - \frac{a_N^2}{N} I_{\text{path}} \left( \left\{ x(t) : 0 \leq t \leq T \right\} \right) \right\}
\]
Comments on the ASEP

For TASEP ($d = 1, p = 1$), $X(t)$ is a Poisson process with rate $1 - \rho$, see [Liggett’85].

For ASEP, the following Poissonian approximation holds:

$$X(t) = N(t) - R(t) + R(0),$$

where $N(t)$ is a Poisson process with rate $(p - q)(1 - \rho)$, and $B(t)$ is a stationary process on $\mathbb{Z}$ satisfying that there exists $\theta > 0$,

$$\mathbb{E}[e^{\theta|R(t)|}] < +\infty$$

uniformly in time $t$.

Thus, for any $\delta > 0$ (recall $\sqrt{N} \ll a_N \ll N$),

$$\frac{N}{a_N^2} \limsup_{N \to \infty} \log \mathbb{P}\left(\frac{|R(tN)|}{a_N} > \delta\right) = -\infty.$$
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Outline of the proof

The main idea is to relate the position of the tagged particle to the empirical measure of the process, and then use MDP from hydrodynamic limits and contraction principle to conclude the proof.

Let $J_{x,x+1}(t)$ be the current across the bound $(x, x+1)$ up to time $t$.

Above, $X(t) = 2$ and $J_{-1,0}(t) = 1$.

For $X(t) > 0$, $J_{-1,0}(t) = \sum_{x=0}^{X(t)-1} \eta_x(t)$, $J_{-1,0}(t) = \sum_{x=0}^{\infty} (\eta_x(t) - \eta_x(0))$. 
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MDP from hydrodynamic limits

For $G \in \mathcal{S}(\mathbb{R})$, define the empirical measure of the SSEP as

$$\langle \mu^N_t, G \rangle = \frac{1}{a_N} \sum_{x \in \mathbb{Z}} (\eta_x(tN^2) - \rho) G(x/N), \quad \sqrt{N} \ll a_N \ll N.$$

For $G \in \mathcal{C}_c^{1,\infty}([0, T] \times \mathbb{R})$ and $\mu \in \mathcal{D}([0, T], \mathcal{S}'(\mathbb{R}))$, define

$$l(\mu, G) = \langle \mu_T, G_T \rangle - \langle \mu_0, G_0 \rangle - \int_0^T \langle \mu_s, (\partial_s + (1/2)\partial_u^2) G_s \rangle \, ds.$$

The rate function $Q = Q_{\text{dyn}} + Q_{\text{ini}}$, where

$$Q_{\text{dyn}}(\mu) = \sup_{G \in \mathcal{C}_c^{1,\infty}([0, T] \times \mathbb{R})} \left\{ l(\mu, G) - \frac{\chi(\rho)}{2} \int_0^T \int_\mathbb{R} (\partial_u G)^2 (s, u) \, duds \right\}$$

$$Q_{\text{ini}}(\mu_0) = \sup_{\phi \in \mathcal{C}_c^\infty(\mathbb{R})} \left\{ \langle \mu_0, \phi \rangle - \frac{\chi(\rho)}{2} \int_\mathbb{R} \phi^2(u) \, du \right\}.$$
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Theorem (Gao-Quastel’03)

Suppose $\eta(0) \sim \nu_\rho$ for $\rho \in (0, 1)$. The sequence of processes $\{\mu^N_t : 0 \leq t \leq T\}_{N \geq 1}$ satisfies the MDP with decay rate $a^2_N/N$ and with rate function $Q(\cdot)$.

See (Kipnis-Olla-Varadhan’89) for LDP from hydrodynamic limits.
MDP for the tagged particle

\[
\frac{1}{a_N} J_{-1,0}(t N^2) = \frac{1}{a_N} \sum_{x=0}^{\infty} \left\{ (\eta_x(t N^2) - \rho) - (\eta_x(0) - \rho) \right\} \\
= \langle \mu_t^N - \mu_0^N, \chi_{[0, +\infty)} \rangle.
\]

\[
\frac{1}{a_N} J_{-1,0}(t N^2) = \frac{1}{a_N} \sum_{x=0}^{X(t N^2)-1} \eta_{t N^2}(x) \\
= \frac{1}{a_N} \sum_{x=0}^{X(t N^2)-1} (\eta_x(t N^2) - \rho) + \frac{\rho}{a_N} X(t N^2),
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The first term on the right hand side is negligible in the rate function:

- If \(|X(tN^2)| > \delta a_N\), by standard large deviation results, it is exponentially small with rate \(a_N^2/N\);
- otherwise, the contribution is \(O(\delta)\), and we let \(\delta \to 0\).

By contraction principle, the rate function for the tagged particle process \(\{X(tN^2), 0 \leq t \leq T\}\) should be

\[ I(x(\cdot)) = \inf \{ Q(\mu) : \langle \mu_t - \mu_0, \chi[0, +\infty) \rangle = \rho x(t), \forall 0 \leq t \leq T \}. \]

Problems
(1) Not easy to apply the contraction principle to the whole sample path.
(2) Not easy to solve the above variational formula.
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1. Not easy to apply the contraction principle to the whole sample path.
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Strategies

1. Prove finite-dimensional MDP for \( \{X(t_i N^2), 1 \leq i \leq n\} \),

\[
I(x(t_i)_{i=1}^n) = \inf \left\{ Q(\mu) : \langle \mu_{t_i} - \mu_0, \chi_{[0, +\infty]} \rangle = \rho x(t_i), 1 \leq i \leq n \right\}
\]

\[
= \frac{1}{2\sigma^2} x \cdot A^{-1} x,
\]

where \( x = (x(t_1), \ldots, x(t_n))^T \) and \( A = (a(t_i, t_j))_{1 \leq i, j \leq n} \),

\[
a(t, s) = \frac{1}{2}(t^{1/2} + s^{1/2} - |t - s|^{1/2}).
\]

2. Prove the process \( \{X(t N^2), 0 \leq t \leq T\} \) is exponentially tight.

3. By standard large deviation results, the process \( \{X(t N^2), 0 \leq t \leq T\} \) satisfies MDP with rate

\[
I(x(\cdot)) = \sup \left\{ \frac{1}{2\sigma^2} x \cdot A^{-1} x : n \geq 1, 0 \leq t_1 < t_2 < \ldots < t_n \leq T, \right.
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t_j \in \Delta^c_x \text{ for all } 1 \leq j \leq n \}.
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Prove finite-dimensional MDP for $\{X(t_i N^2), 1 \leq i \leq n\}$,

$$I(x(t_i)^n_{i=1}) = \inf \left\{ Q(\mu) : \langle \mu t_i - \mu_0, \chi_{[0,+\infty)} \rangle = \rho x(t_i), 1 \leq i \leq n \right\}$$

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Thanks!

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