Moderate deviation principles for the tagged particle in the simple exclusion process

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The exclusion process

The state space is $\{0,1\}^{\mathbb{Z}^d}$. An element of the state space is called a configuration, denoted by η. For $x \in \mathbb{Z}^d$, $\eta_x \in \{0,1\}$ is the number of particles at site x.

Generator of the process $\eta(t)$: for local functions f on $\{0,1\}^{\mathbb{Z}^d}$,

$$Lf(\eta) = \sum_{x,y\in\mathbb{Z}^d} p(y-x)\eta_x(1-\eta_y)[f(\eta^{x,y} - f(\eta))],$$

where $\eta_z^{x,y} = \eta_x$ for $z = y$, $= \eta_y$ for $z = x$, and $= \eta_z$ otherwise.

Figure: SSEP: $p = q = 1/2$. ASEP: $p = 1 - q \in (1/2, 1]$.

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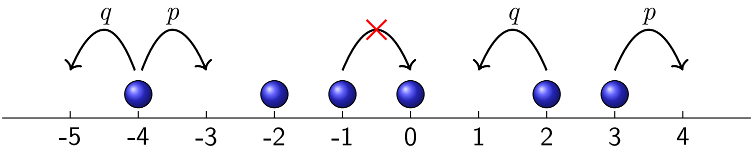


Figure: SSEP: $p = q = 1/2$. ASEP: $p = 1 - q \in (1/2, 1]$.

For $\rho \in [0,1]$, let ν_ρ be the product measure on $\{0,1\}^{\mathbb{Z}^d}$ with marginals

$$\nu_\rho(\eta_x = 1) = \rho, \quad x \in \mathbb{Z}^d.$$

It is well known that ν_ρ is invariant for the exclusion process, see (Liggett'85 and '99).

Check directly that

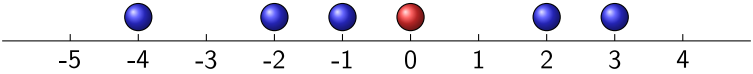
$$\int Lf d\nu_\rho = 0.$$

The tagged particle

Let the initial measure of the process be

$$\nu_\rho^*(\cdot) = \nu_\rho(\cdot | \eta_0 = 1).$$

Denote by $X(t)$ the position of the tagged particle at time t. The process $X(t)$ is not Markovian!



Define the environmental process seen from the tagged particle as

$$\zeta_x(t) = \eta_{X_t + x}(t), \quad x \in \mathbb{Z}^d.$$

Since the process $\eta(t)$ is translation invariant, the process $\zeta(t)$ is Markovian. Moreover, ν_ρ^* is invariant for the process $\zeta(t)$.

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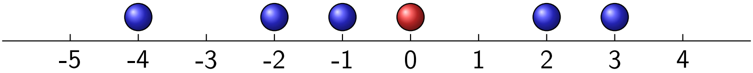

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Related work

Assume $p(\cdot)$ has finite range and the process $\eta(t)$ starts from the initial measure ν_ρ^*.

Law of large numbers

$$\lim_{t\to\infty} \frac{X(t)}{t} = (1-\rho)\sum_{x\in\mathbb{Z}^d} xp(x) \quad \text{almost surely,}$$

see (Saada'87).

Central limit theorems

- For $d=1, p(1)=p(-1)=1/2$,

$$\frac{X_{tN^2}}{N^{1/2}} \Rightarrow fBM(1/4), \quad N\to+\infty.$$

See (Arratia'83) (De Masi-Ferrari'02) (Peligrad-Sethuraman'08).

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$$\frac{X_{tN^2}}{N} \Rightarrow BM, \quad N \to +\infty.$$

See (Kipnis-Varadhan'86) for the symmetric case, (Varadhan'95) mean zero case, (Kipnis'86) ASEP, (Sethuraman-Varadhan-Yau'00) asymmetric case in dimension $d \geq 3$, (Komorowski-Landim-Olla'12).

- For asymmetric case in $d \leq 2$ except ASEP, CLT and inriance principles are open, see (Sethuraman'06).

Large deviations

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What is MDP?

Let $X_1, X_2, \ldots$ be independent random variables, and $S_N = \sum_{i=1}^{N} X_i$.

Law of large numbers.

$$S_N/N \to \mu := E[X_1] \quad \text{almost surely.}$$

Central limit theorems.

$$(S_N - N\mu)/N^{1/2} \Rightarrow N(0, \sigma^2).$$

Large deviations.

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Moderate deviations. For $N^{1/2} \ll a_N \ll N$,

$$\log \mathbb{P}\Big(\frac{S_N - N\mu}{a_N} = x\Big) \approx -\frac{a_N^2}{N}\frac{x^2}{2\sigma^2}.$$

- First intuition:

$$\mathbb{P}\Big(\frac{S_N - N\mu}{a_N} = x\Big) \approx \mathbb{P}\Big(N(0,\sigma^2) = \frac{a_N x}{\sqrt{N}}\Big) \approx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big\{-\frac{a_N^2}{N}\frac{x^2}{2\sigma^2}\Big\}.$$

- Second intuition:

$$\mathbb{P}\Big(\frac{S_N - N\mu}{a_N} = x\Big) = \mathbb{P}\Big(\frac{S_N}{N} = \mu + \frac{a_N}{N}x\Big) \approx \exp\Big\{-NI\Big(\mu + \frac{a_N}{N}x\Big)\Big\}.$$

Since $I(\mu) = I'(\mu) = 0$,

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One-point MDP for the tagged particle

Consider the SSEP, that is, $d = 1$ and $p(1) = p(-1) = 1/2$. Recall

$$X(t)/t^{1/4} \Rightarrow N(0, \sigma^2), \quad \sigma^2 = \sqrt{2/\pi}(1-\rho)/\rho.$$

Fix $\sqrt{N} \ll a_N \ll N$ and $T > 0$. Define

$$I(\alpha) = -\alpha^2/(2\sqrt{T}\sigma^2), \quad \alpha \in \mathbb{R}.$$

Theorem (Xue-Z.'23)

The sequence $\{X(TN^2)/a_N\}_{N\geq 1}$ satisfies the MDP with decay rate a_N^2/N and with rate function $I(\cdot)$. Precisely speaking, for any closed set $C \subset \mathbb{R}$ and for any open set $O \in \mathbb{R}$,

$$\limsup_{N\to\infty} \frac{N}{a_N^2} \log \mathbb{P}\big(X(TN^2)/a_N \in C\big) \leq -\inf_{\alpha\in C} I(\alpha),$$

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Sample path MDP for the tagged particle

The fractional Brownian motion $\{B_{1/4}(t) : t \geq 0\}$ is a Gaussian process with covariance

$$\mathrm{Cov}(B_{1/4}(t), B_{1/4}(s)) = \frac{1}{2}(t^{1/2} + s^{1/2} - |t-s|^{1/2}).$$

It also has the following representation

$$B_{1/4}(t) = \int_0^t K(t,s)\,dB(s).$$

Let $\mathcal{H}$ be the set of càdlàg functions $f\colon [0,T] \to \mathbb{R}$ such that there exists a function $h_f \in L^2([0,T])$ satisfying

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For any càdlàg functions $f\colon [0, T] \to \mathbb{R}$, define

$$I_{\mathrm{path}}(f) = \begin{cases} \frac{1}{2}\int_0^T h_f(s)^2\, ds, & \text{if } f \in \mathcal{H}; \\ +\infty, & \text{otherwise.} \end{cases}$$

I_{path} is the large deviation rate function of the sequence of processes $\{B_{1/4}(t)/\sqrt{N} : t \geq 0\}_{N\geq 1}$.

Assume

$$\sqrt{N \log N} \ll a_N \ll N.$$

Theorem (Xue-Z.'23)

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Intuitive explanation

$$\begin{aligned}
&\mathbb{P}\Big(\{X(tN^2)/a_N : 0 \le t \le T\} = \{x(t) : 0 \le t \le T\}\Big) \\
&= \mathbb{P}\Big(\Big\{\frac{X(tN^2)}{\sqrt{N}} : 0 \le t \le T\Big\} = \Big\{\frac{a_N}{\sqrt{N}} x(t) : 0 \le t \le T\Big\}\Big) \\
&\approx \mathbb{P}\Big(\Big\{\frac{\sqrt{N}}{a_N} B_{1/4}(t) : 0 \le t \le T\Big\} = \Big\{x(t) : 0 \le t \le T\Big\}\Big) \\
&\approx \exp\Big\{-\frac{a_N^2}{N} I_{\text{path}}(\{x(t) : 0 \le t \le T\})\Big\}
\end{aligned}$$

Comments on the ASEP

For TASEP ($d = 1, p = 1$), $X(t)$ is a Poisson process with rate $1 - \rho$, see (Liggett'85).

For ASEP, the following Poissonian approximation holds:

$$X(t) = N(t) - R(t) + R(0),$$

where $N(t)$ is a Poisson process with rate $(p - q)(1 - \rho)$, and $B(t)$ is a stationary process on $\mathbb{Z}$ satisfying that there exists $\theta > 0$,

$$\mathbb{E}\Big[e^{\theta|R(t)|}\Big] < +\infty$$

uniformly in time t.
Thus, for any $\delta > 0$ (recall $\sqrt{N} \ll a_N \ll N$),

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Outline of the proof

The main idea is to relate the position of the tagged particle to the empirical measure of the process, and then use MDP from hydrodynamic limits and contraction principle to conclude the proof.

Let $J_{x,x+1}(t)$ be the current across the bound $(x, x+1)$ up to time t.
Above, $X(t) = 2$ and $J_{-1,0}(t) = 1$.
For $X(t) > 0$,

$$J_{-1,0}(t) = \sum_{x=0}^{X(t)-1} \eta_x(t), \quad J_{-1,0}(t) = \sum_{x=0}^{\infty} (\eta_x(t) - \eta_x(0)).$$

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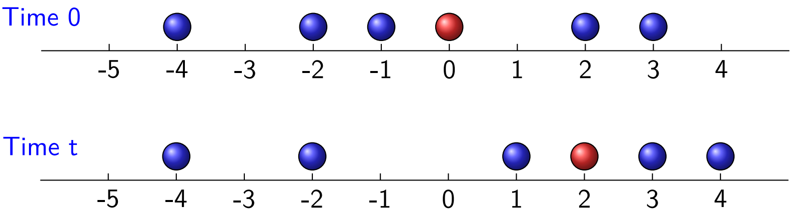

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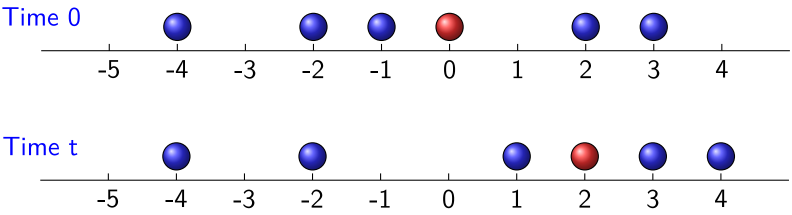

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MDP from hydrodynamic limits

For $G \in \mathcal{S}(\mathbb{R})$, define the empirical measure of the SSEP as

$$\langle \mu_t^N, G \rangle = \frac{1}{a_N} \sum_{x \in \mathbb{Z}} (\eta_x(tN^2) - \rho) G(x/N), \quad \sqrt{N} \ll a_N \ll N.$$

For $G \in \mathcal{C}_c^{1,\infty}([0,T] \times \mathbb{R})$ and $\mu \in D([0,T], \mathcal{S}'(\mathbb{R}))$, define

$$l(\mu, G) = \langle \mu_T, G_T \rangle - \langle \mu_0, G_0 \rangle - \int_0^T \left\langle \mu_s, \left(\partial_s + (1/2)\partial_u^2\right) G_s \right\rangle ds.$$

The rate function $\mathcal{Q} = \mathcal{Q}_{\mathrm{dyn}} + \mathcal{Q}_{\mathrm{ini}}$, where

$$\mathcal{Q}_{\mathrm{dyn}}(\mu) = \sup_{G \in \mathcal{C}_c^{1,\infty}([0,T]\times\mathbb{R})} \left\{ l(\mu, G) - \frac{\chi(\rho)}{2} \int_0^T \int_{\mathbb{R}} \left(\partial_u G\right)^2 (s,u)\, du ds \right\}$$

$$\mathcal{Q}_{\mathrm{ini}}(\mu_0) = \sup_{\phi \in \mathcal{C}_c^\infty(\mathbb{R})} \left\{ \langle \mu_0, \phi \rangle - \frac{\chi(\rho)}{2} \int_{\mathbb{R}} \phi^2(u)\, du \right\}$$

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Theorem (Gao-Quastel'03)

Suppose $\eta(0) \sim \nu_\rho$ for $\rho \in (0,1)$. The sequence of processes $\{\mu_t^N : 0 \leq t \leq T\}_{N \geq 1}$ satisfies the MDP with decay rate a_N^2/N and with rate function $\mathcal{Q}(\cdot)$.

See (Kipnis-Olla-Varadhan'89) for LDP from hydrodynamic limits.

MDP for the tagged particle

$$\frac{1}{a_N} J_{-1,0}(tN^2) = \frac{1}{a_N} \sum_{x=0}^{\infty} \left\{ (\eta_x(tN^2) - \rho) - (\eta_x(0) - \rho) \right\}$$
$$= \langle \mu_t^N - \mu_0^N, \chi_{[0,+\infty)} \rangle.$$

$$\frac{1}{a_N} J_{-1,0}\left(tN^2\right) = \frac{1}{a_N} \sum_{x=0}^{X\left(tN^2\right)-1} \eta_{tN^2}(x)$$
$$= \frac{1}{a_N} \sum_{x=0}^{X\left(tN^2\right)-1} \left(\eta_x(tN^2) - \rho\right) + \frac{\rho}{a_N} X\left(tN^2\right),$$

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The first term on the right hand side is negligible in the rate function:

- If $|X(tN^2)| > \delta a_N$, by standard large deviation results, it is exponentially small with rate a_N^2/N;
- otherwise, the contribution is $O(\delta)$, and we let $\delta \to 0$.

By contraction principle, the rate function for the tagged particle process $\{X(tN^2), 0 \le t \le T\}$ should be

$$I(x(\cdot)) = \inf \big\{ \mathcal{Q}(\mu) : \langle \mu_t - \mu_0, \chi_{[0,+\infty)} \rangle = \rho x(t), \forall 0 \le t \le T \big\}.$$

Problems

(1) Not easy to apply the contraction principle to the whole sample path.
(2) Not easy to solve the above variational formula.

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Strategies

1. Prove finite-dimensional MDP for $\{X(t_iN^2), 1 \le i \le n\}$,

$$I(x(t_i)_{i=1}^n) = \inf\left\{\mathcal{Q}(\mu) : \langle \mu_{t_i} - \mu_0, \chi_{[0,+\infty)}\rangle = \rho x(t_i), 1 \le i \le n\right\}$$
$$= \frac{1}{2\sigma^2}\mathbf{x}\cdot A^{-1}\mathbf{x},$$

where $\mathbf{x} = (x(t_1), \dots, x(t_n))^T$ and $A = (a(t_i, t_j))_{1\le i,j\le n}$,

$$a(t,s) = \frac{1}{2}(t^{1/2} + s^{1/2} - |t-s|^{1/2}).$$

2. Prove the process $\{X(tN^2), 0 \le t \le T\}$ is exponentially tight.
3. By standard large deviation results, the process $\{X(tN^2), 0 \le t \le T\}$ satisfies MDP with rate

$$I(x(\cdot)) = \sup\left\{\frac{1}{2\sigma^2}\mathbf{x}\cdot A^{-1}\mathbf{x} : n \ge 1, 0 \le t_1 < t_2 < \dots < t_n \le T, \right.$$
$$\left. t_j \in \Delta_x^c \text{ for all } 1 \le j \le n\right\}.$$

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Thanks!

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