Equilibrium perturbations for stochastic interacting systems

Linjie Zhao[1] **(joint work with Lu Xu**[2]**)**

[1]Huazhong University of Science and Technology, China
[2]Gran Sasso Science Institute, Italy

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The exclusion process

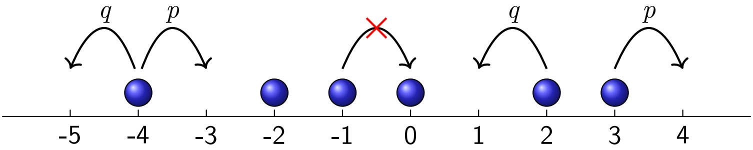

Figure: ASEP: $p = 1 - q \in (1/2, 1]$.

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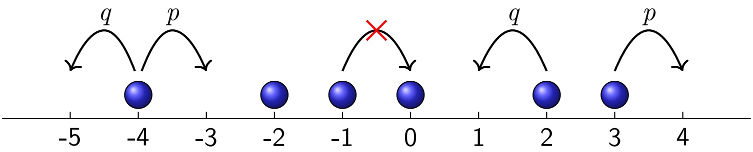

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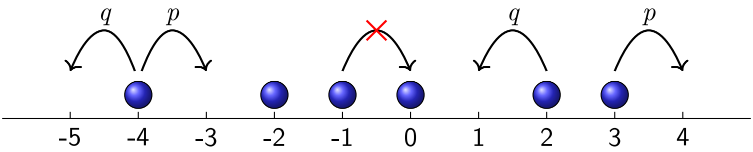

Figure: ASEP: $p = 1 - q \in (1/2, 1]$.

The configuration space is $\Omega := \{0,1\}^{\mathbb{Z}^d}$. For any $\eta \in \Omega$ and $x \in \mathbb{Z}^d$, $\eta_x \in \{0,1\}$ is the number of particles at site x.
Generator of the process $\eta(t)$: for local functions f on Ω,

$$Lf(\eta) = \sum_{x,y \in \mathbb{Z}^d} p(y-x)\eta_x(1-\eta_y)[f(\eta^{x,y}) - f(\eta)],$$

where $\eta_z^{x,y} = \eta_x$ for $z = y$, $= \eta_y$ for $z = x$, and $= \eta_z$ otherwise.

Hydrodynamic limits

Assume $m := \sum_{x\in\mathbb{Z}^d} xp(x) \neq 0$. Define the empirical measure of the process

$$\pi_t^N(du) = \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \eta_x(tN)\delta_{x/N}(du).$$

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Under mild conditions,

$$\lim_{N\to\infty} \pi_t^N(du) = \rho(t,u)du \quad \text{in probability,}$$

where the hydrodynamic equation is

$$\partial_t\rho(t,u) + m\cdot\nabla f(\rho(t,u)) = 0$$

with initial condition ρ_{ini}. Above, $f(\rho) = \rho(1-\rho)$.

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- Relative entropy method only in the smooth regime [Yau'1991];
- Attractiveness method [Rezakhanlou'1991].

$$\eta(0) \leq \zeta(0) \Rightarrow \eta(t) \leq \zeta(t).$$

Navier-Stokes corrections

One expects an order of N^{-1} correction to the hydrodynamic equation [Page 185, Kipnis-Landim'1998]:

$$\partial_t \rho^N + m \cdot \nabla f(\rho^N) = \frac{1}{N} \sum_{i,j=1}^{d} \partial_{u_i} \big[D_{i,j}(\rho^N) \partial_{u_j} \rho^N \big].$$

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- First order correction. Define

$$q^N(t,u) := \mathbb{E}[\eta_{[uN]}(tN)].$$

Then, under suitable initial conditions,

$$\lim_{N \to \infty} N[q^N - \rho^N] = 0$$

in a weak sense.
Asymmetric EP in $d \geq 3$ [Landim-Olla-Yau'1997, CPAM],
EP with speed change in $d \geq 3$ [Janvresse'1998, AoP].

- Long time behavior. If

$$\partial_t \rho(t,u) + m \cdot \nabla f(\rho(t,u)) = 0,$$

then the entropy solution converges to a stationary solution which is constant along the drift:

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Thus, under diffusive scaling, one expects that

$$m \cdot \nabla \lim_{N\to\infty} \mathbb{E}[\eta_{[uN]}(tN^2)] = 0,$$

and that on the hyperplane orthogonal to the drift, the profile obeys a parabolic equation.
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- The incompressible limit...

Equilibrium perturbation for the PDE

For $d \geq 3$, one expects an order of N^{-1} correction to the hydrodynamic equation: (the fluctuation has order $N^{-d/2}$)

$$\partial_t \rho^N + m \cdot \nabla f(\rho^N) = \frac{1}{N} \sum_{i,j=1}^{d} \partial_{u_i} \big[D_{i,j}(\rho^N) \partial_{u_j} \rho^N \big].$$

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For $\rho_* \in [0,1]$, consider

$$\rho^N(t,u) = \rho_* + \frac{1}{N^\alpha} a^N\Big(\frac{t}{N^\gamma}, u\Big).$$

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Then,

$$\partial_t \rho^N(t,u) = \frac{1}{N^{\gamma+\alpha}} a^N(\tfrac{t}{N^\gamma}, u), \qquad D_{i,j}(\rho_N(t,u)) = D_{i,j}(\rho_*) + O(N^{-\alpha}),$$
$$f(\rho^N(t,u)) = f(\rho_*) + \frac{f'(\rho_*)}{N^\alpha} a^N(\tfrac{t}{N^\gamma}, u) + \frac{f''(\rho_*)}{2N^\alpha} [a^N(\tfrac{t}{N^\gamma}, u)]^2 + O(N^{-3\alpha}).$$

Inserting the above Taylor expansions to the corrected hydrodynamic equation,

$$\partial_t a^N + N^\gamma f'(\rho_*) m \cdot \nabla a^N + \frac{1}{2} N^{\gamma-\alpha} f''(\rho_*) m \cdot \nabla[(a^N)^2]$$
$$= N^{\gamma-1} \sum_{i,j=1}^{d} D_{i,j}(\rho_*) \partial^2_{u_i,u_j} a^N + O(N^{\gamma-\alpha-1} + N^{\gamma-2\alpha}).$$

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The blue term can be removed by a Galilean transformation

$$a^N(t,u) = b^N(t, u - mtN^\gamma f'(\rho_*)).$$

Finally,

$$\partial_t b^N + \frac{1}{2} N^{\gamma-\alpha} f''(\rho_*) m \cdot \nabla[(b^N)^2]$$
$$= N^{\gamma-1} \sum_{i,j=1}^{d} D_{i,j}(\rho_*) \partial^2_{u_i,u_j} b^N + O(N^{\gamma-\alpha-1} + N^{\gamma-2\alpha}).$$

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Define $b(t,u) = \lim_{N\to\infty} b^N(t,u)$.

- If $\gamma = \alpha = 1$,

$$\partial_t b + \frac{1}{2} f''(\rho_*) m \cdot \nabla b^2 = \sum_{i,j=1}^{d} D_{i,j}(\rho_*) \partial^2_{u_i,u_j} b\,;$$

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- If $\gamma = \alpha < 1$,

$$\partial_t b + \frac{1}{2} f''(\rho_*) m \cdot \nabla b^2 = 0\,;$$

$$\partial_t b^N + \frac{1}{2} N^{\gamma-\alpha} f''(\rho_*) m \cdot \nabla [(b^N)^2] = N^{\gamma-1} \sum_{i,j=1}^{d} D_{i,j}(\rho_*) \partial^2_{u_i,u_j} b^N + O(N^{\gamma-\alpha-1} + N^{\gamma-2\alpha}).$$

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- If $\gamma = \alpha < 1$,

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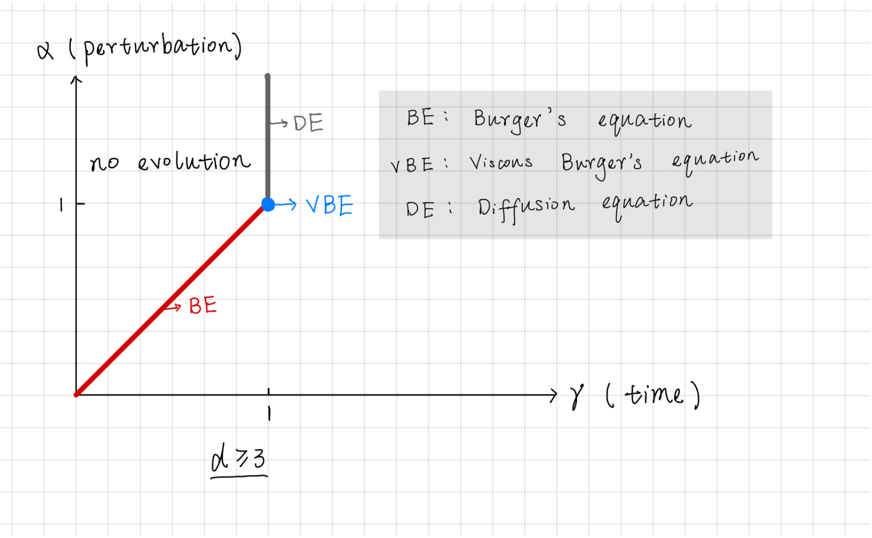
- If $\gamma = 1 < \alpha$,

$$\partial_t b = \sum_{i,j=1}^{d} D_{i,j}(\rho_*) \partial^2_{u_i,u_j} b.$$

Phase transition in $d \geq 3$



Equilibrium perturbation for the IPS

By hydrodynamic limits theory,

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For $u\in\mathbb{R}^d$, take

$$H(v)=H_u(v)=(2\varepsilon)^{-d}\mathbf{1}\{|u-v|\le\varepsilon\},$$

then

$$\frac{1}{(2\varepsilon N)^d}\sum_{|x-uN|\le\varepsilon N}\eta_x(tN)\approx\rho^N(t,u)=\rho_*+\frac{1}{N^\alpha}b^N\Big(\tfrac{t}{N^\gamma},u-mtf'(\rho_*)\Big).$$

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Thus,

$$b^N(t,u)\approx\frac{N^\alpha}{(2\varepsilon N)^d}\sum_{|x-(u+mtN^\gamma f'(\rho_*))N|\le\varepsilon N}(\eta_x(tN^{1+\gamma})-\rho_*).$$

$$b^N(t,u) \approx \frac{N^\alpha}{(2\varepsilon N)^d} \sum_{|x-(u+mtN^\gamma f'(\rho_*))N| \leq \varepsilon N} (\eta_x(tN^{1+\gamma}) - \rho_*).$$

Therefore, under mild conditions, we expect that for any test function H,

$$\lim_{N\to\infty} \frac{1}{N^{d-\alpha}} \sum_{x\in\mathbb{Z}^d} \big(\eta_x(tN^{1+\gamma}) - \rho_*\big) H\big(\tfrac{x}{N} - mtN^\gamma f'(\rho_*)\big)$$

$$= \int_{\mathbb{R}^d} b(t,u)H(u)du \quad \text{in probability.}$$

Related literature

- Asymmetric EP in $d \geq 3$ [Esposito-Marra-Yau'1994, Rev. Math. Phys.]: $\gamma = \alpha = 1$,

$$\lim_{N \to \infty} \frac{1}{N^{d-1}} \sum_{x \in \mathbb{Z}^d} \big(\eta_x(tN^2) - \rho_*\big) H\big(\tfrac{x}{N} - mtNf'(\rho_*)\big) = \int_{\mathbb{R}^d} b(t,u) H(u)\, du \quad \text{in probability,}$$

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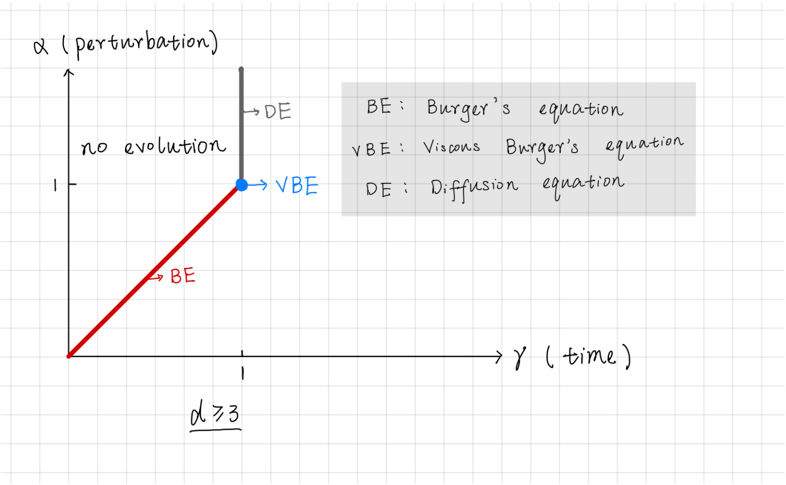
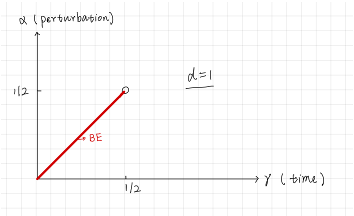
The correct time scaling for $d = 1$ should be $N^{3/2}$, while for $d = 2$ there is a logarithmic correction to N^2.


α (perturbation)
no evolution
DE
VBE
BE
1
1
γ (time)
d ≥ 3
BE : Burger's equation
VBE : Viscous Burger's equation
DE : Diffusion equation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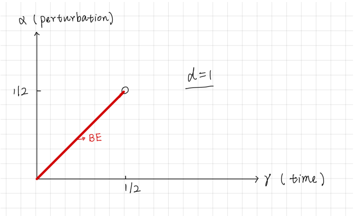

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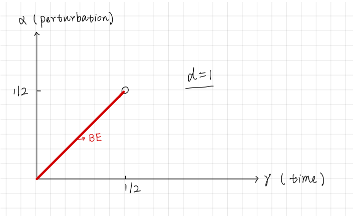

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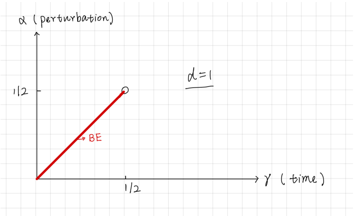

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- Deposition models in $d = 1$ [Tóth-Valkó'2002, JSP]: $\gamma = \alpha < 1/5$. Relative entropy method, only in the smooth regime.
- Weakly asymmetric EP in $d \geq 1$ [Jara-Landim-Tsunoda'2021, AIHP]: for $\gamma = 1$ and under some constraints on α, the limit is viscous Burgers equation. Improved relative entropy method by Jara and Menezes.

Generalized exclusion process

The state space is $\Omega = \{0, 1, \ldots, K\}^{\mathbb{T}_N^d}$. For $\eta \in \Omega$, $\eta_x \in \{0, 1, \ldots, K\}$ is the number of particles at site x. For $1 \leq i \leq d$, a particle jumps from site x to site $x \pm e_i$ at rate

$$p_i \eta_x (K - \eta_{x+e_i}) \quad \text{and} \quad q_i \eta_x (K - \eta_{x-e_i}).$$

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$$m = (m_i)_{1 \leq i \leq d} = (p_i - q_i)_{1 \leq i \leq d}.$$

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The process has a family of product invariant measures ν_ρ^N on Ω indexed by particle density $\rho \in [0, K]$,

$$\nu_\rho^N(\eta_x = k) = \binom{K}{k} \left(\frac{\rho}{K}\right)^k \left(1 - \frac{\rho}{K}\right)^{K-k}, \quad k = 0, 1, \ldots, K.$$

For any profile $\rho : \mathbb{T}^d \to [0, K]$, define $\nu^N_{\rho(\cdot)}$ as the product measure on Ω with marginals

$$\nu^N_{\rho(\cdot)}(\eta_x = k) = \nu^N_{\rho(x/N)}(\eta_x = k), \quad x \in \mathbb{T}^d_N,\ k = 1, \dots, K.$$

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Define the reference profile

$$\rho^N(t, u) = \rho_* + N^{-\alpha} b(tN^{\gamma-\alpha}, u - mtN^{\gamma} f'(\rho_*)),$$

where, $f(\rho) = \rho(K - \rho)$, and

$$\partial_t b = m \cdot \nabla b^2.$$

Let $\nu^N_t = \nu^N_{\rho^N(t,\cdot)}$ be the reference measure.

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Let $\nu^N_t = \nu^N_{\rho^N(t,\cdot)}$ be the reference measure.
Recall the relative entropy is defined as

$$H(\mu|\nu) = \int f \log f \, d\nu, \quad f = \frac{d\mu}{d\nu}.$$

Main results [Xu-Z.'2023, EJP]

Let μ_t^N be the distribution of the process at time $tN^{1+\gamma}$. If

$$H(\mu_0^N|\nu_0^N) = o(N^{d-2\alpha}),$$

then under some constraints on γ and α, for any $t > 0$ such that $b(t,u)$ is smooth,

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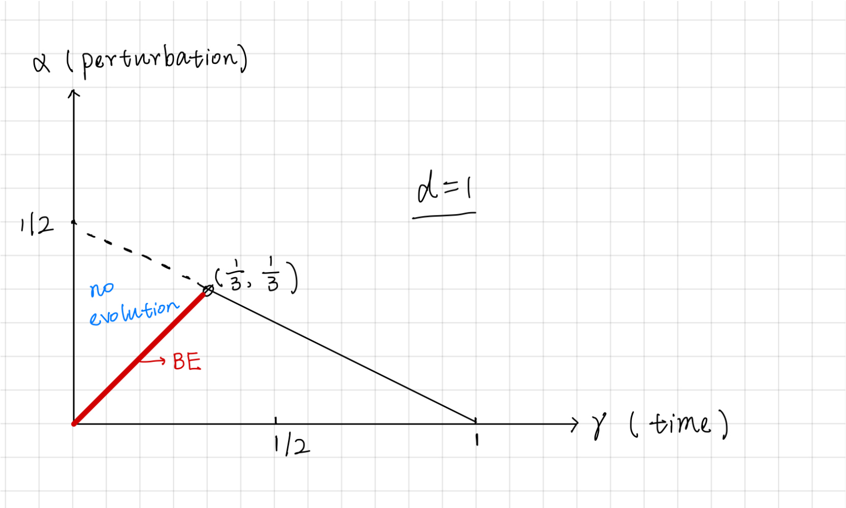
As a corollary, for any $t > 0$ and test function H,

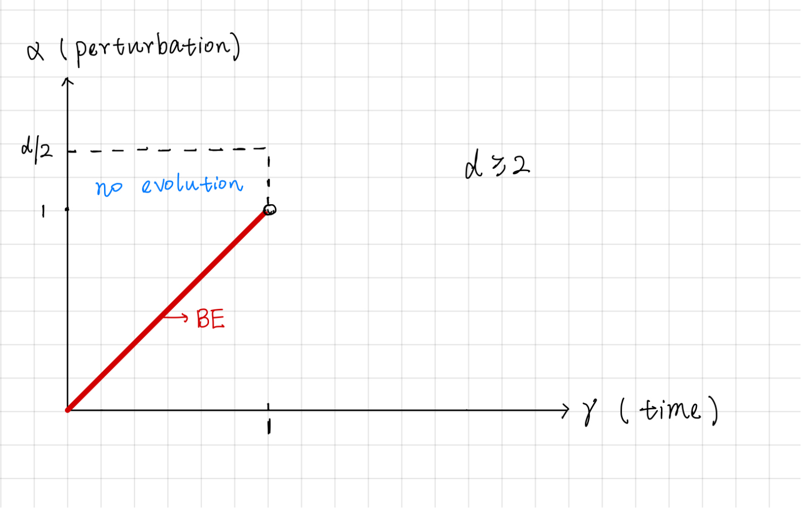
$$\lim_{N\to\infty} \frac{1}{N^{d-\alpha}} \sum_{x\in\mathbb{T}_N^d} (\eta_x(tN^{1+\gamma}) - \rho_*) H(\tfrac{x}{N} - mf'(\rho_*)tN^\gamma)$$

$$= \int_{\mathbb{T}^d} \tilde{b}(t,u) H(u)\, du \quad \text{in probability.}$$

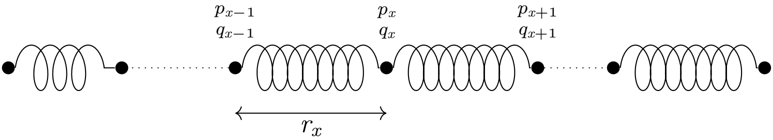
$d = 1$



$d \geq 2$



1-d chain of anharmonic oscillators



Above, $p_x =$ momentum of the particle x; $q_x =$ position of the particle x; $r_x = q_x - q_{x-1}$ the inter-particle distance. Assume $V \in \mathcal{C}^2(\mathbb{R})$ with bounded second derivative. We consider the periodic case $x \in \mathbb{T}_N$.

$$
\begin{aligned}
dp_x(t) &= \left[V'(r_{x+1}(t)) - V'(r_x(t)) \right] dt, \\
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\end{aligned}
$$

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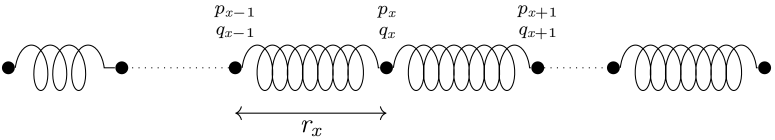

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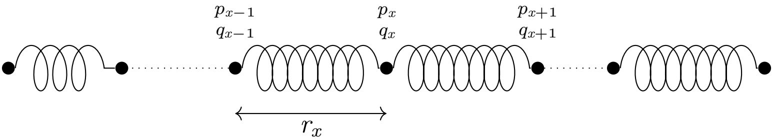
Momentum $\sum p_x$, volume $\sum r_x$ and energy $\sum [V(r_x) + p_x^2/2]$ are conserved.

1-d chain of anharmonic oscillators



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&\quad + \frac{\beta \kappa_N}{2} \big[V'(r_{x+1}(t)) + V'(r_{x-1}(t)) - 2 V'(r_x(t)) \big] \, dt \\
&\quad + \sqrt{\kappa_N} \big[dB_t^{x-1} - dB_t^x \big].
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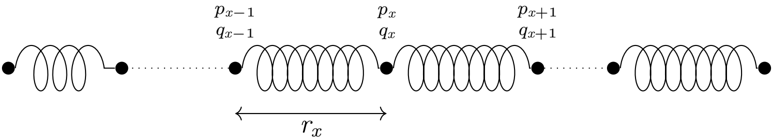

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Only momentum $\sum p_x$ and volume $\sum r_x$ are conserved.

The hydrodynamic equation is given by the following p-system

$$\partial_t \mathfrak{p} = \partial_u \boldsymbol{\tau}(\mathfrak{r}), \quad \partial_t \mathfrak{r} = \partial_u \mathfrak{p}.$$

Above, $\mathfrak{p} = \mathfrak{p}(t,u), \mathfrak{r} = \mathfrak{r}(t,u)$ for $u \in \mathbb{T}$, and $\boldsymbol{\tau} = \boldsymbol{\tau}(\mathfrak{r})$ is the equilibrium tension.

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The current of the system is

$$J(\mathfrak{p},\mathfrak{r}) = (-\boldsymbol{\tau}(\mathfrak{r}), -\mathfrak{p}).$$

Let

$$A = \begin{pmatrix} \partial_{\mathfrak{p}} J^{\mathfrak{p}} & \partial_{\mathfrak{r}} J^{\mathfrak{p}} \\ \partial_{\mathfrak{p}} J^{\mathfrak{r}} & \partial_{\mathfrak{r}} J^{\mathfrak{r}} \end{pmatrix} \Big|_{(\mathfrak{p}_*,\mathfrak{r}_*)} = \begin{pmatrix} 0 & -\boldsymbol{\tau}'(\mathfrak{r}_*) \\ -1 & 0 \end{pmatrix}$$

The matrix A has two eigenvalues $\pm\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}$ with corresponding right eigenvalues

$$\mathbf{v}_+ := \begin{pmatrix} -\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)} \\ 1 \end{pmatrix} \quad \mathbf{v}_- := \begin{pmatrix} \sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)} \\ 1 \end{pmatrix}$$

Main results [Xu-Z.'2023, EJP]

Assume initially, for any test function H,

$$\lim_{N\to\infty} \frac{1}{N^{1-\alpha}} \sum_{x\in\mathbb{T}_N} \begin{pmatrix} p_x(0) - \mathfrak{p}_* \\ r_x(0) - \mathfrak{r}_* \end{pmatrix} H\Big(\frac{x}{N}\Big) = \sum_{j=\pm} \mathbf{v}_j \int_{\mathbb{T}} \sigma_j^{\rm ini}(u) H(u)\,du \quad \text{in probability,}$$

where $\sigma_\pm^{\rm ini}$ are the initial profiles satisfying

$$\int_{\mathbb{T}} \sigma_+^{\rm ini}\,du = \int_{\mathbb{T}} \sigma_-^{\rm ini}\,du = 0.$$

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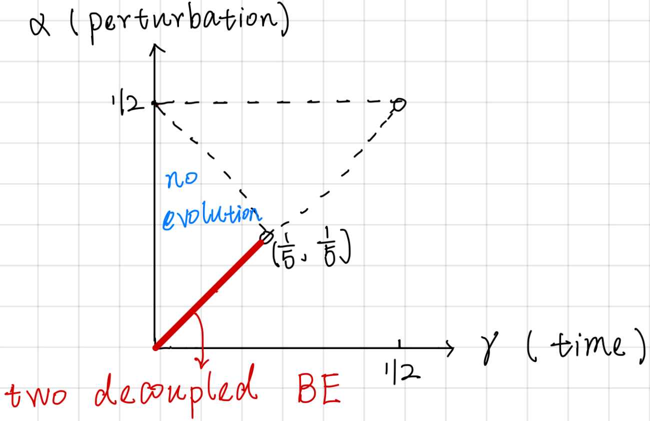
We observe the evolution of perturbed conserved quantities along the characteristic lines.


α (perturbation)
1/2
no evolution
(1/5, 1/5)
γ (time)
1/2
two decoupled BE

Related literature

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- It is always non-resonant for 1-d systems with two conservation laws, which is not the case for systems with three or more conservation laws.
- EP with collisions: $d \geq 3$ [Esposito-Marra-Yau'1996, CMP], weakly asymmetric [Meurs-Tsunoda-Xu'2024, arXiv:2402.10375].

Thanks!

Email: linjie_zhao@hust.edu.cn